Continuity and End Behavior

A function is said to be continuous when it has no breaks in its graph. Another way to think of this - a function is continuous if you can draw it without lifting your pencil.

If there is a break in the graph the function is discontinuous at that point.

There are three different kinds of discontinuity.

1. **Infinite Discontinuity** (or asymptotic discontinuity) occurs at asymptotes (invisible lines in a graph that the graph approaches but never touches or crosses). Asymptotes occur at points where the denominator equals zero. (When U factor the denominator if you still have x's then you'll have asymptotes where they zero out).

2. **Jump Discontinuity** can occur when you graph piecewise functions. Jump discontinuities occur where the function approaches two different values from either side of the discontinuity. Piecewise functions are defined piece-by-piece and their graph seems to "jump" from one branch to the other.

3. **Point discontinuity (or Removable Discontinuity)** occurs when functions are defined differently for a certain point. They are discontinuous at that single point. Point discontinuities occur when a function is defined specifically for an isolated x-value. (In other words where the graph will have a hole and the point will be defined elsewhere - see below graph a). Point discontinuities also occur when we can cancel a term in the denominator and the numerator when we reduce - (see below graph b).

**Note:** $x^2 - 4$ factors to be $(x - 2)(x + 2)$ so you now have the function $\frac{(x - 2)(x + 2)}{x - 2}$ and the x-2 terms cancel out, BUT before that there WAS a place in the graph where the denominator would have equaled 0.

$$x - 2 = 0$$

$$x = 2$$
This graph has an example of each of the three types of discontinuity

a = Jump @ $x = -1$

b = Infinite @ $x = 0$

c = Point @ $x = 1$

Give it a try: Scroll to Ex 2 from the below link

Example 2 Recognizing Points of Discontinuity in a Graph

Let $f$ have the graph shown below.

Q f is discontinuous at $x = -2$  
Q f is Undefined at $x = -1$  
Q f is Undefined at $x = 1$  
Q f is Continuous at $x = 2$  

Point C Jumps  
Infinite  
No Break in Graph
Limit: the concept of approaching a value without ever reaching it.

\[ \lim_{x \to c} f(x) = L \]

read: The limit of \( f(x) \) as \( x \) approaches \( c \) is \( L \).

The **right limit** of \( f(x) \) as \( x \) approaches \( a \):

\[ \lim_{x \to a^+} f(x) \]

is the value the function \( f(x) \) approaches, if any, as \( x \) values larger than \( a \) get close to \( a \).

The **left limit** of \( f(x) \) as \( x \) approaches \( a \):

\[ \lim_{x \to a^-} f(x) \]

is the value the function \( f(x) \) approaches, if any, as \( x \) values smaller than \( a \) get close to \( a \).

The **limit** of \( f(x) \) as \( x \) approaches \( a \):

\[ \lim_{x \to a} f(x) = L \]

if both the right limit and the left limit values exist and they are equal to \( L \).

An important fact is that the value of \( \lim_{x \to a} f(x) \) is independent of \( f(a) \). That is, the limit depends on the values of the function near \( x = a \) but not \( f(a) \) itself. Also, \( a \) can be \( +\infty \) or \( -\infty \) and the value of a limit may also be \( +\infty \) or \( -\infty \).

Cited from:
http://faculty.weber.edu/ptalaga/math1100/LAB8.pdf

For graphs of functions with a removable discontinuity, the limit of \( f(x) \) at point \( c \) exists, but either the value of the function at \( c \) is undefined or the value of \( f(c) \) is not the same as the value of the limit at point \( c \).

Infinite and jump discontinuities are classified as **nonremovable discontinuities**. A nonremovable discontinuity can’t be eliminated by redefining the function at that point, since the function approaches different values from the left and right sides at that point or does not approach a single value at all. Instead it’s increasing or decreasing indefinitely.

Which all leads to the test for the continuity of a function....
End Behavior: describes how a function behaves at both of its ends. In other words it describes what the values of $f(x)$ does as $x$ increases and as $x$ decreases.

**Trick:** if the ends of the graph point up or down then the value of $f(x)$ will approach positive/negative infinity. **BUT** if the ends of the graph points left/right then the value of $f(x)$ will approach an actual number.

Limits are one way to describe end behavior. \[ \lim_{x \to +\infty} f(x) \quad \text{or} \quad \lim_{x \to -\infty} f(x) \]

Ex. \( g(x) = x^3 - 9x + 2 \)

\[ \lim_{x \to +\infty} g(x) = +\infty \]  
\[ \lim_{x \to -\infty} g(x) = -\infty \]

Ex. \( h(x) = \frac{x}{x^2 - 2x + 8} \)

\[ \lim_{x \to +\infty} g(x) = 0 \]  
\[ \lim_{x \to -\infty} g(x) = 0 \]
Your Turn. Determine if the function is continuous at the point. If discontinuous, identify the type.

A) \( f(x) = \sqrt{x-4} \) at \( x = 5 \)
1. Does \( f(5) \) exist? 
   \( \lim_{x \to 5} f(x) = 4.58 \) 
   \( f(5) \) exists.

2. \( \lim_{x \to 5^-} f(x) \) exists? 
   \( \lim_{x \to 5^-} f(x) = 4.58 \)
   \( \lim_{x \to 5^+} f(x) = 4.58 \)
   \( \lim_{x \to 5} f(x) \) exists.

3. Does \( \lim_{x \to 5} f(x) = f(5) \)? 
   Yes.

B) \( h(x) = \frac{x^2 - 32}{x+4} \) at \( x = -4 \)

1. Does \( f(-4) \) exist? 
   No.

Now, what kind is it?

\[
\begin{array}{c|cccccccc}
-6 & -5.2 & -5.1 & -5 & -5.0 & -5.01 & -5.001 & -5.0001 & -5.00001 \\
-4 & -1.2 & -1.1 & -1 & -1.01 & -1.001 & -1.0001 & -1.00001 \\
-2 & -0.2 & -0.1 & -0.001 & -0.0001 & -0.00001 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

\( \lim_{x \to -4^-} f(x) = 12 \) 
\( \lim_{x \to -4^+} f(x) = -1 \) 
\lim_{x \to -4} f(x) \) exists.

\( \lim_{x \to -4} f(x) = \lim_{x \to -4} f(x) \) but \( \lim_{x \to -4} f(x) \) doesn't exist.

= point discontinuity

(p24)
\[ \lim_{x \to 1^+} f(x) = 0.25 \]
\[ \lim_{x \to 1^-} f(x) = -1 \]
\[ \lim_{x \to -\infty} f(x) = -1 \]
\[ \lim_{x \to +\infty} f(x) = +\infty \]
\[ \lim_{x \to 1} f(x) = \text{undefined} \]
\[ f(1) = -1 \]

\[ \lim_{x \to +\infty} f(x) = +\infty \]
\[ \lim_{x \to -\infty} f(x) = +\infty \]

\[ \lim_{x \to 0^+} f(x) = 0 \]
\[ \lim_{x \to 0^-} f(x) = -2 \]
\[ f(0) = 0 \]
\[ \lim_{x \to 0} f(x) = \text{undefined} \]

\[ \lim_{x \to -\infty} f(x) = +\infty \]
\[ \lim_{x \to +\infty} f(x) = -2 \]
\[
\lim_{x \to -\infty} f(x) = -\infty \\
\lim_{x \to +\infty} f(x) = +\infty \\
\lim_{x \to 0^+} f(x) = 0 \\
\lim_{x \to 0^-} f(x) = 0 \\
\lim_{x \to 0} f(x) = 0
\]

\[
f(2) = 2.5 \\
f(-2) = 3 \\
f(0) = 0
\]

\[
\lim_{x \to \infty} f(x) = 4 \\
\lim_{x \to -\infty} f(x) = 4
\]

\[
f(0) = 0 \\
f(5) = \text{undefined}
\]

\[
f(1) = 3
\]

\[
\lim_{x \to 1^-} f(x) = 3 \\
\lim_{x \to 1^+} f(x) = 4 \\
\lim_{x \to 1} f(x) = \text{undefined}
\]